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The Potential Role of Manufacturers' Orders in the Linear-Quadratic Flexible Accelerator Model

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Abstract

The linear-quadratic flexible accelerator model has been a staple for empirical analysis within the inventories literature. A key assumption of the model is that sales are a proxy for demand, thus inventories are generated in instances where production does not equilibrate with demand. We seek to improve upon the benchmark linear-quadratic model by introducing firm orders, thus allowing for differentiation between realized and expected demand. Estimation results suggest that the omission of orders heavily biases the coefficient sign and magnitude associated with sales. Furthermore, the estimated adjustment speed of orders is both larger in magnitude and statistical significance than for sales. The disparity in the rates of adjustment between expected and realized demand provides a new contribution towards understanding the adjustment speed puzzles pervasive in the literature. Finally, the addition of orders provides stronger evidence of a cointegrating relationship in a trivariate system of equations versus a standard bivariate system containing strictly inventories and sales. However, exogenous order shocks do not seem to meaningfully impact inventory investment, nor sales. Furthermore, forecast error variance for sales, and inventories are not well explained by orders. These results suggest that orders are important in explaining the underlying data generating processes for sales, and inventories, as well as the long-run relationship between inventory, and sales, but offer little in forecasting potential.

Keywords: inventories, M3 data, persistence, cointegration, linear-quadratic

1. Introduction

The linear-quadratic (LQ) model is the empirical workhorse of the inventories literature. Originally described in Holt (1960), the LQ model characterizes a cost-minimizing firm's inventory investment decision. The LQ model has been extended through works like Ramey (1989), Blanchard (1983), M. Lovell (1961), M.C. Lovell (1962), West (1986), Humphreys et al. (2001) and Iacoviello et al. (2011). Beyond journal publications, economic handbook chapters like Ramey & West (1999) and West (1993), and surveys like Blinder & Maccini (1991) provide an exhaustive synthesis of the inventories literature and the LQ model's history. Despite the maturity of inventories research, pervasive inventory puzzles described in Maccini et al. (2015) and Wen (2011) fuel new works within the literature alongside growing interest in global value chains and supply chain distortions. Furthermore, on average from 1997 through the present day, the dollar value of all inventories held by manufacturers and retailers as a share of real GDP hovers around 11.3%.(Note 1) Given the sheer volume of inventories held by the private sector, understanding firm incentives for holding inventories in spite of their cost burdens, and their potential relationship to the business cycle are of importance researchers, firm managers, consumers, forecasters, and policymakers alike.

By design, the LQ model is intended to capture key stylized facts about inventory behavior and generalize the mechanisms driving a representative firm’s motivations to hold inventories. More importantly, however, the LQ model serves to relate micro-level behaviors of firms to macro-level business cycle fluctuations. As noted in Ramey & West (1999), at a microeconomic level, inventories tend to be a stabilizing force through smoothing production in environments of fluctuating demand; however, conversely, at a macroeconomic level, inventories tend to be destabilizing via the accelerator principle.

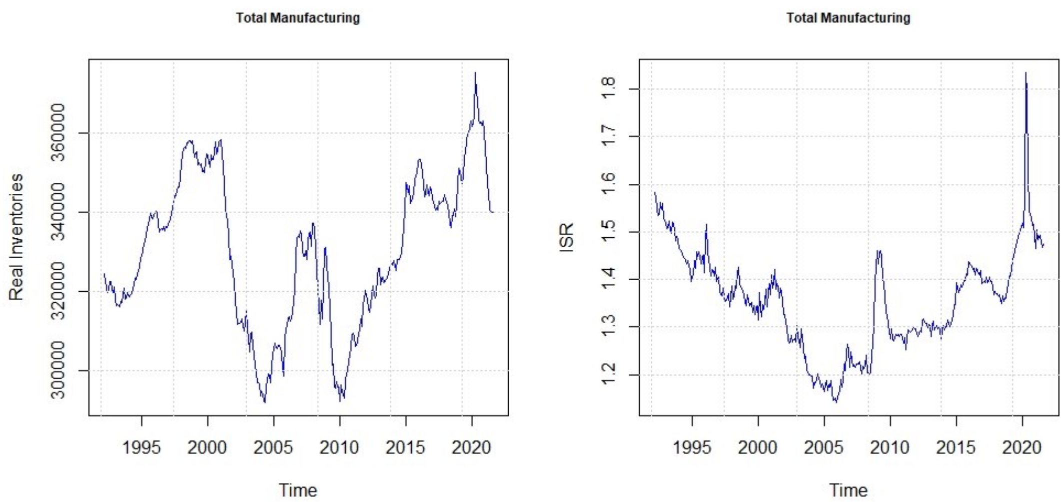

Figure 1. Inventories, and the Inventory-to-Sales Ratio

These polarized facts introduce a core paradox at the heart of the inventories literature: inventories are a stabilizing force at the micro-level and destabilizing force at the macro-level simultaneously. Further highlighted in Ramey & West (1999), inventories move in a procyclical manner, while the inventory-to-sales ratio is countercyclical and highly persistent. Figure 1 illustrates these stylized facts depicting real manufacturing inventories over time and the real manufacturing inventory-to-sales ratio.

The validity of these stylized facts seems to hold up until the Financial Crisis and subsequent Great Trade Collapse described in Bems et al. (2013) who argue that during the Great Recession, the collapse in aggregate expenditures was largely concentrated in durable goods leading to an amplification of firm inventory adjustments contrary to the norm. We can relate these stylized facts to a more technical framework described by seminal pieces like Ramey & West (1999), West (1993) and Hamilton (2002). Specifically, these pieces aim to use the LQ model to define a firm's optimum inventory investment decision, ΔH_t, in a reduced form, which is expressed as: $\Delta H_t = A_0 + A_1 H_{t-1} + A_2 S_{t-1} + \epsilon_t$. (Note 2)

Traditionally, the equation for inventory investment is used in combination with a cointegrating equation and the data generating process (DGP) for sales, S_t, to produce a vector error correction model (VECM) from which adjustment speeds can be measured. Notwithstanding the system of equations that can be estimated from the LQ model, the inventory investment equation on its own is quite telling. In particular, the literature predicts that A_1 should be negative and highly significant to capture the high degree of persistence in H_{t-1}. More interestingly, however, the model would predict A_2 to be positive and significant. In the retail setting, $S_{t-1} > 0$ is sensible as it would imply that as firms observe increasing sales, they accelerate their inventory positions to meeting heightened demand. However, as stressed in Blinder & Maccini (1991), and reinforced by Blanchard (1983), M. Lovell (1961) and M.C. Lovell (1962) among many others, most inventories lie in the manufacturing sector, thus S_t from a practical standpoint is more akin to shipments, rather than sales strictly speaking. (Note 3)

Beyond these points, as noted in the operations research and supply chain management fields, there is very often lead-time between when an order is placed, a good is produced, and then finally fulfilled in the form of a shipment. Typically, new orders trigger the production of new goods at which point they go through the production process and are eventually shipped. This gap of time between the start of production and fulfillment prevalent in the manufacturing sector poses potential complications for the traditional solution to the LQ model. For instance, if one cannot distinguish between newly placed orders and shipments, it is likely A_2 will be heavily biased absorbing both the effects of realized demand and expected demand.

Furthermore, as highlighted by Maccini et al. (2015) and Wen (2011), the cointegrating relationship between inventories and sales has been considerably weaker in recent decades despite the assumption of cointegration underlying the LQ model's foundations put forth in Hamilton (2002). On the other hand, if the cointegrating relationship between inventory and sales has deteriorated, then why is the inventory-to-sales ratio (ISR) still above unity and highly persistent? Furthermore, why do we still see inventories move in a procyclical manner with the business cycle along with sales? In a bivariate VECM, we would expect the stochastic trend shared between inventory and sales to have also have a long-run equilibrium that both inventories and sales should adjust to if there is short-run disequilibrium. Again, we posit that perhaps there is a significant omission in the underlying DGP or sets of DGP's in the LQ model. Specifically, we believe that one must distinguish between orders and shipments in manufacturing to potentially rectify these issues.

Beyond the history of the LQ model within the field of economics, both orders and inventories of substantial interest to practitioners within industry for forecasting macroeconomic conditions. (Note 4) Furthermore, while not widely discussed within the economics literature, operations researchers have done well to point out the inefficiencies in global supply chains that can arise from order volatility as highlighted in the seminal work of Lee et al. (1997). Lee et al. (1997) highlight that the phenomena known as the Bullwhip Effect can have dire consequences to firm and industry performance through excess (or inadequate) inventory investment, loss of sales, misguided capacity plans, and forecast inaccuracies.

In particular, the Bullwhip Effect manifests itself as a higher relative volatility of new manufacturer orders relative to sales. Given the reliance and growing interest in the economics of highly integrated value chains today, and in the wake of Covid-19's disruption to global supply chains, the LQ model with some modifications provides a lens through which economists and forecasters can begin to view supply chain distortions that is still structurally consistent its classic microfoundations and features.(Note 5) Put simply, the realities, and growing attention to global value chains provides new opportunities for research on inventories, and its foundational models to better explain wider supply chain phenomena, and demand volatility, particularly in the manufacturing sector.

2. Model

Consider a variation of the LQ model described by Hamilton (2002) and Herrera (2018). The representative firm maximizes expected profit subject to a budget constraint with prices, P_t, taken as exogenous. Our goal is to solve for inventory investment, ΔH_t, such that it can be estimated in a reduced form as both a single-equation model, and as a system of equations alongside the data generating process for shipments, S_t, and orders, O_t. The firm's objective function is described by equation (1).

$$\underset{(Q_t, H_t)}{\text{Max}} \left\{ E_0 \sum_{t=0}^{\infty} \beta^t (P_t S_t - C_t) \right\} \tag{1}$$

This objective function is further constrained by equations (2), and (3), respectively.

$$C_t = \frac{1}{2}\{a_0(\Delta Q_t)^2 + a_1[(Q_t - U_{ct})^2 + a_2(H_{t-1} - a_4 - a_3 S_t)^2]\} \tag{2}$$

$$Q_t = S_t + \Delta H_t \tag{3}$$

Finally, to close out the model's setup, the following laws of motion for sales, and orders, respectively, by equations (4), and (5).

$$S_t = A_s + \gamma_1 S_{t-1} + \gamma_2 O_{t-1} + e_{st} \tag{4}$$

$$O_t = A_o + \omega_1 O_{t-1} + e_{ot} \tag{5}$$

The first-order condition of our objective function with respect to H_t yields equation (6).

$$E_t\,[a_0(\Delta Q_t - 2\,\beta\Delta Q_{t+1} + \beta^2\,\Delta Q_{t+2}) + a_1(Q_t - U_{ct}) - \beta a_1(Q_{t+1} - U_{ct+1}) + \beta a_1\,a_2(H_t - a_4 - a_3\,S_{t+1})] = 0 \tag{6}$$

Under the assumption that adjustment costs are equal to zero ($a_0 = 0$), and dividing by a_1, we arrive at equation (7).

$$E_t[(Q_t - U_{ct}) - \beta(Q_{t+1} - U_{ct+1}) + \beta a_2(H_t - a_4 - a_3 S_{t+1})] = 0 \tag{7}$$

Assuming that S_t, and U_{ct} are integrated at an order of one, or simply I (1), and by leveraging our production identity, and sales generating process, described by equations (3), and (4), respectively, we can simply equation (7) further (by implication of S_t, and U_{ct} being I(1), we have $U_{ct} = S_t + v_{ct}$). Beyond this, we can save some additional algebra by recognizing that v_{ct+1}, and e_{st+1} are both $\sim \mathrm{IID}(0, \sigma^2)$, which implies that $E_t e_{ct+1} = E_t e_{st+1} = 0$. With all this in mind, we arrive at equation (8) after some manipulation (8).

$$E_t[\Delta H_t - v_{ct} - \beta\Delta H_{t+1} + \beta a_2(H_t - a_4 - a_3 A_s - a_3\gamma_1 S_t - a_3\gamma_2 O_t)] = 0 \tag{8}$$

To simplify further, let $-a_4 - a_3 A_s = \phi_0$, $-a_3\gamma_1 = \phi_1$, and $-a_3\gamma_2 = \phi_2$. This gives us equation (9).

$$E_t[\Delta H_t - v_{ct} - \beta\Delta H_{t+1} + \beta a_2(H_t + \phi_0 + \phi_1 S_t + \phi_2 O_t)] = 0 \tag{9}$$

From the above equation, we can define our cointegrating equation as $w_t = H_t + \phi_0 + \phi_1 S_t + \phi_2 O_t$. Note that we assume there is only one cointegrating relationship (long-run relationship) shared between all three variables in our equation. We will confirm this later with data.

Via substitution, we can rewrite equation (9) by plugging in our cointegrating equation and arrive at: $E_t[\Delta H_t - \beta\Delta H_{t+1} + \beta a_2 w_t - v_{ct}] = 0$. It is further worth noting that $\Delta H_t = \Delta w_t - \phi_1\Delta S_t - \phi_2\Delta O_t$. As a result of this, we can further simplify our result. After collecting all terms common to w_t, and rearranging, we arrive at equation (10), which can be simplified further to equation (11).

$$E_t[(1 + \beta + \beta a_2)w_t - w_{t-1} - \beta w_{t+1}] = E_t[\phi_1\Delta S_t + \phi_1\Delta S_{t+1} + \phi_2\Delta O_t + \phi_2\Delta O_{t+1} + v_{ct}] \tag{10}$$

$$E_t[(1 + \beta + \beta a_2)w_t - w_{t-1} - \beta w_{t+1}] = E_t[(\phi_1 - \beta)(\Delta S_t + \Delta S_{t+1}) + (\phi_2 - \beta)(\Delta O_t + \Delta O_{t+1}) + v_{ct}] \tag{11}$$

Take note that $S_t - S_{t-1} = \Delta S_t = \frac{A_s + \gamma_2 O_{t-1} + e_{st}}{\gamma_1}$ and $O_t - O_{t-1} = \Delta O_t = \frac{A_o + e_{ot}}{\omega_1}$. Let $k_s = A_s/\gamma_1, \Gamma_s = \gamma_1/\gamma_2$, and $\epsilon_{st} = e_{st}/\gamma_1$. Furthermore, define $\Delta S_t = k_s + \Gamma_s O_t + \epsilon_{st}$, and $\Delta O_t = k_o + \epsilon_{ot}$. By forward iteration, $\Delta S_{t+1} = k_s + \Gamma_s O_{t+1} + \epsilon_{st+1}$ and $O_{t+1} = k_o + \epsilon_{ot+1}$. As a result, $\Delta O_t + \Delta O_{t+1} = 2k_o + \epsilon_{ot} + \epsilon_{ot+1}$. Let $x_{st} = 2k_s + \epsilon_{st}$ and $x_{ot} = 2k_o + \epsilon_{ot}$. Note that x_{st} and x_{ot} are both stochastic. Recall that $E_t\epsilon_{ot+1} = E_t\epsilon_{st+1} = 0$. With these definitions, and some rearrangement, equation (11) can be rewritten as equation (12).

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$$\begin{aligned} &E_t[(1+\beta+\beta a_2)w_t - w_{t-1} - \beta w_{t+1}] \\ &= E_t[(\phi_1-\beta)(\Gamma_s k_o + \Gamma_s \epsilon_{ot} + x_{st}) + (\phi_2-\beta)x_{ot} + v_{ct}] \end{aligned} \tag{12}$$

Let $(\phi_1-\beta)(\Gamma_s k_o) = h_0$, $(\phi_1-\beta)(\Gamma_s x_{st}) = h_{1t}$, and $(\phi_2-\beta)X_{ot} + v_{ct} = h_{2t}$, where both h_{1t}, and h_{2t} are accordingly stochastic. Finally, let $h_{3t} = h_{1t} + h_{2t}$. These simplifications produce equation (13).

$$E_t[(1+\beta+\beta a_2)w_t - w_{t-1} - \beta w_{t+1}] = h_0 + h_{3t} \tag{13}$$

Following Hamilton (2002) and Sargent (2009), we use the lag operator, L, and some algebraic manipulation to rewrite equation (13) as $E_t\beta\left[\left(1-\frac{1+\beta+\beta a_2}{\beta}L+\beta^{-1}L^2\right)w_{t+1}\right] = -h_0 - h_{3t}$. This equation can be factorized as a well-known difference equation described Hamilton (2002), and Sargent (2009). Equation (14) captures this difference equation and has real roots $0 < \lambda_1 < 1$, $\lambda_2 = 1/(\beta\lambda_1) > 1$.

$$(1-\lambda_1 z)(1-\lambda_2 z) = \left(1-\frac{1+\beta+\beta a_2}{\beta}z+\beta^{-1}z^2\right) \tag{14}$$

Recalling that h_{3t} is our white noise term, we can arrive at (15).

$$w_t = \lambda_1 w_{t-1} + \frac{\gamma_1 h_0}{1-\lambda_2^{-1}} + \lambda_1 h_{3t} \tag{15}$$

It's worth highlighting that equation (15) is a stationary AR(1) process. Recalling that $w_t = H_t + \phi_0 + \phi_1 S_t + \phi_2 O_t$, we can add H_{t-1} to both sides of equation (15), thus allowing it to be expressed as $w_t = \Delta H_t + (H_{t-1} + \phi_0 + \phi_1 S_t + \phi_2 O_t)$. If we define$w_{t-1} = H_{t-1} + \phi_0 + \phi_1 S_{t-1} + \phi_2 O_{t-1}$, then via substitution, and some rearrangement, we arrive at equation (16).

$$\begin{aligned} \Delta H_t = \lambda_1(H_{t-1}+\phi_0+\phi_1 S_{t-1}+\phi_2 O_{t-1}) + (\gamma_1 h_0)/(1-\lambda_2^{-1}) + \lambda_1 h_{3t} - H_{t-1} - \phi_0 \\ -\phi_1(A_s + \gamma_1 S_{t-1} + \gamma_2 O_{t-1} + e_{st}) - \phi_2(A_o + \omega_1 O_{t-1} + e_{ot}) \end{aligned} \tag{16}$$

After collecting all terms common to H_{t-1}, O_{t-1}, and S_{t-1}, we can rearrange equation (16) as equation (17).

$$\begin{aligned} \Delta H_t = (\lambda_1-1)H_{t-1} + (\lambda_1\phi_1 - \phi_1\gamma_1)S_{t-1} + (\lambda_1\phi_2 - \phi_1\gamma_2 - \phi_2\omega_1)O_{t-1} \\ +(\gamma_1 h_0)/(1-\lambda_2^{-1}) + \lambda_1 h_{3t} + \lambda_1\phi_0 - \phi_0 - \phi_1 A_s - \phi_1 e_{st} - \phi_2 A_o - \phi_2 e_{ot} \end{aligned} \tag{17}$$

Finally, we can collect all constant, and stochastic terms to produce our reduced form equation, described by (18).

$$\Delta H_t = B_0 + B_1 H_{t-1} + B_2 S_{t-1} + B_3 O_{t-1} + \varepsilon_t \tag{18}$$

Where $B_1 = (\lambda_1 - 1)$, $B_2 = (\lambda_1\phi_1 - \phi_1\gamma_1)$, and $B_3 = (\lambda_1\phi_2 - \phi_1\gamma_2 - \phi_2\omega_1)$. Our constant is defined as $B_0 = (\gamma_1 h_0)/(1-\lambda_2^{-1}) + \lambda_1\phi_0 - \phi_0 - \phi_1 A_s - \phi_2 A_o$, and our stochastic term is defined as $\varepsilon_t = \lambda_1 h_{3t} - \phi_1 e_{st} - \phi_2 e_{ot}$.

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3. Empirical Evidence

Our orders-augmented LQ model in a reduced form setting can be estimated as a single equation model via equation (18), or as a system of equations using equations using (18), (4), and (5) in combination with our cointegrating equation, $w_t = H_t + \phi_0 + \phi_1 S_t + \phi_2 O_t$. We present single and simultaneous equation estimation results for total manufacturing and compare them with baseline estimations using the traditional solution to the LQ model. Our main results show that the inclusion of orders into the solution for optimal inventory investment changes both the sign and significance of the lagged sales coefficient from positive to negative for the total manufacturing sector.

Furthermore, results from a system-wise estimation procedure for total manufacturing uncover that orders adjust faster and at a level of higher significance than sales or inventories to restore equilibrium, thus adding a new finding the pervasive adjustment speed puzzle in the literature. Finally, cointegration tests show significantly stronger evidence of one cointegration relationship in a trivariate system versus a bivariate equation system.

3.1 Data

Our data source is the M3 Manufacturing Survey published by the US Census Bureau. All data are in monthly buckets ranging from March 1992 through August 2022 and deflated using the CPI index.(Note 6) Our data is specific to total manufacturing. All data are in terms of millions of US dollars. Figure 2 provides visual evidence of the data.

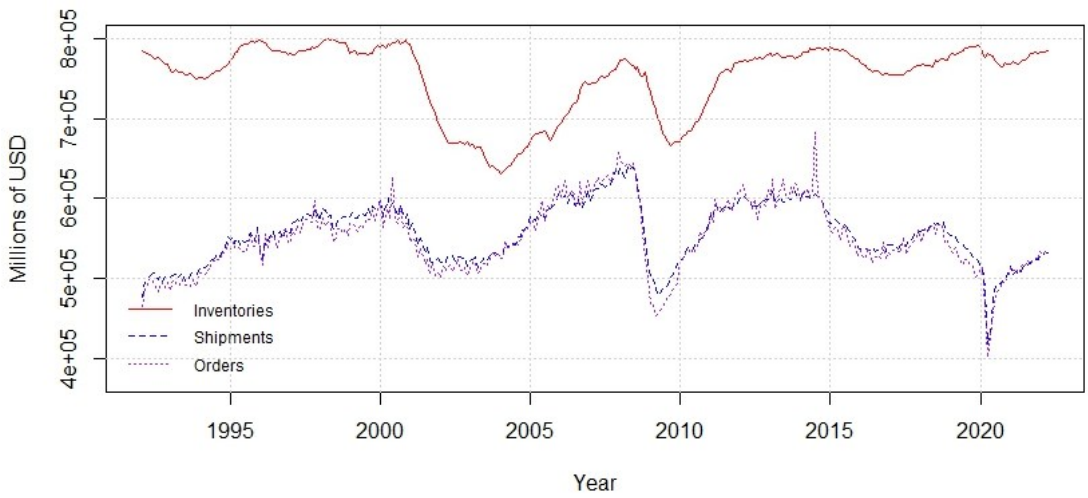

Figure 2. M3 Manufacturing Data: Total Manufacturing

A key insight from Figure 2 is that while orders and sales tend to comove with very close correspondence, they are seldom identical. This statistical observation wherein expected demand fails to equate with realized demand reaffirms our motivation and the inclusion of O_t in equations (4), and (5). Other key descriptive statistics relating to inventory, sales, orders, and production (where production is $Q_t = S_t + \Delta H_t$), including correlation, are described in Table 1 below.

Table 1. Descriptive Statistics

Statistic	H_t	S_t	O_t	Q_t	ΔH_t	ΔS_t	ΔO_t	ΔQ_t
Mean (μ)	1,108,690	816,663.2	808,624.5	852,749.5	-2,343.074	-1,272.532	-1,205.192	-1,440.673
Std. Dev. (σ)	235,576	158,066.9	155,918.4	168,667.5	6,788.181	11,936.18	20,077.14	15,014.59
Minimum	786,052	471,732.2	452,792.5	520,587.9	-27,003.24	-72,946.25	-98,307.4	-62,151.34
Maximum	1,634,245	1,060,463	1,067,265	1,143,247	23,962.15	47,510.7	88,991.63	47,432.57
$cor(H_t)$	1	--	--	--	-0.14	0.02	0.01	0.03
$cor(S_t)$	0.93	1	--	--	-0.06	0.07	0.04	0.10
$cor(O_t)$	0.91	0.99	1	--	-0.04	0.09	0.09	0.13
$cor(Q_t)$	0.93	0.96	0.95	1	-0.17	0.00	-0.01	0.05
$cor(\Delta H_t)$	-0.14	0.02	0.01	0.03	1	--	--	--
$cor(\Delta S_t)$	-0.06	0.07	0.04	0.10	0.02	1	--	--
$cor(\Delta O_t)$	-0.04	0.09	0.09	0.13	0.04	0.66	1	--
$cor(\Delta Q_t)$	-0.17	0.00	-0.01	0.05	0.49	0.76	0.53	1

Beyond Table 1, it's also worth reporting and discussing the average inventory-to-sales (shipments) ratio, and the volatility of orders relative to shipments. Over our full sample, we report a mean inventories-to-sales ratio, H_t/S_t, of 1.36. This number is surprisingly high in an era of vendor-managed inventories, and just-in-time (JIT) inventory practices. In a sense, a high inventory-to-sales ratio in this instance illustrates that firms are still bearing considerable costs of "hoarding" inventories, which can in-part be attributable to the need to buffer supply chain disruptions, and unexpected demand distortions.(Note 7)

As for the relative volatility of orders relative to shipments, $var(O_t)/var(S_t)$, we report a variance ratio of 0.97. This variance ratio is known as the Bullwhip Effect, or demand amplification. Interestingly, a Bullwhip Effect < 1 suggests there is little demand amplification at play in the manufacturing sector, despite the inventory-to-sales ratio being $>$ 1. This result tells us that over most of our sample, firms are relatively successful at smoothing production volatility relative to demand volatility but are engaging in inventory hoarding or using inventories as a buffer to shocks as though they were not smoothing production.

3.2 Single-Equation Results

From equation (18), we have an estimable expression for optimal inventory growth based on the firm's optimization procedure described in our setup. We compare estimation results of equation (18) to a model without orders, which would be the conventional solution to this model described by Holt (1960), and expanded upon by Hamilton (2002), Herrera (2018), and Ramey & West (1999) among others. Table 2 illustrates these results.

We note significant statistical differences between both reduced form models. Firstly, both B_0 and lagged inventories, H_{t-1}, are statistically zero in our new model, but carry marginal statistical significance in the baseline model. Lagged sales, S_{t-1}, is statistically significant in both models, but differently signed. The interpretation from the baseline model is that a one-unit change in lagged sales causes inventories to grow by 0.03 or $30,000. In our updated model, a unit change in S_{t-1} causes ΔH_t to fall by 0.05 or $50,000. Furthermore, a unit increase in lagged orders, O_{t-1}, causes ΔH_t to increase by 0.08 or $80,000.

Table 2. Single Equation Models

Dependent Variable: ΔH_t				
Coefficient	Equation (18)		Baseline Equation	
B_0	-5819.73	(4121.89)	-10616.62**	(4000.11)
B_1	-0.01	(0.00)	-0.01*	(0.00)
B_2	-0.05**	(0.02)	0.03***	(0.01)
B_3	0.08***	(0.02)		
N	362		362	
Adj. R^2	0.13		0.09	

Note: $*** p < 0.001; ** p < 0.01; * p < 0.05; . p < 0.10$

Quite simply, the omission of orders significantly biases the coefficient for sales. One explanation for this stem directly from the idiosyncrasies of the manufacturing sector. To reiterate aspects of Blinder & Maccini (1991), most inventories lie in the manufacturing sector, wherein orders are generated well in advance of goods fulfillment compared to the retail sector, thus the timing of demand signal generation and production relative to fulfillment allows for inventory investment to arise via a mismatch between anticipated and actual demand.

3.3 Simultaneous-Equation Results

We estimate a VECM to observe changes in adjustment speeds between a bivariate system without orders and a trivariate VECM with orders. We first must test for cointegration. We apply the methodology from Johansen (1995) to identify the presence of cointegration across our trivariate system, $\mathrm{Z_t} = [\mathrm{H_t}, \mathrm{S_t}, \mathrm{O_t}]^T$. These results are reported in Table 3.

Table 3. Cointegration Test Results

Hypothesis	Trace Statistic	10% Crit. Val.	5% Crit. Val.	1% Crit. Val.
$r = 0$	42.81**	32.00	34.91	41.07
$r \leq 1$	20.55*	17.85	19.96	24.60
$r \leq 2$	2.50	7.52	9.24	12.97

Note: $*** p < 0.001; ** p < 0.01; * p < 0.05; . p < 0.10$

Consistent with our model's structural assumptions, there is evidence of, at most, one cointegrating relationship. To proceed further, we must also discern if our data is $I(1)$. We formally test for this using Augmented Dickey-Fuller (ADF) tests. These results are reported in Table 4.

Table 4. ADF Test Results

Variable	T. Statistic	P-Value
H_t	-2.9551	0.1738
S_t	-3.2773	0.07507.
O_t	-3.515	0.0414*
ΔH_t	-4.479	≤0.001***
ΔS_t	-5.618	≤0.001***
ΔO_t	-5.9361	≤0.001***
Note: $*** p < 0.001; ** p < 0.01; * p < 0.05; . p < 0.10$		

We observe that there's weak evidence of a unit root in S_t and O_t. However, there is strong rejection of the presence of a unit root when we difference our data, thus it is more suggestive that our data is $I(1)$, rather than $I(0)$. We proceed to estimate a VECM described by equation (19).

$$\Delta Z_t = A_0 + \Pi Z_{t-1} + \sum_{i=1}^{6} \Omega_i \Delta Z_{t-i} + \eta_t \tag{19}$$

The results are reported in Table 5.

We observe that inventories exhibit high persistence across all three equations in our system, as expected. Shipments are only persistent in the orders equation and are of weak significance in the inventories equation. Finally, orders are movements between both the sales equation and its own equation.

Inventories exhibit an adjustment speed that is statistically zero; however, both shipments and orders adjust slowly at a high level of statistical significance. It seems that across all lags, inventories lead orders more so, but orders lead sales persistently across all six lags. On the other hand, sales seem to drive some significant and persistent effects across both orders and inventories.

More interestingly, in a single equation setting, the level of orders leads inventory investment, but in a VECM setting, inventory growth leads order growth. Moreover, it is orders that adjust to restore equilibrium in the VECM setting despite not leading inventory growth. Finally, it's worth noting that orders load consistently, and positively on shipments, but at relatively low levels (akin to a moving average function). These results at first glance imply that forecasters and firms ought to give more credence to shipments, rather than orders for the sake of forecasting inventory investment positions, and demand.

Table 5. VECM Results

	VECM Equations					
Coefficient	ΔH_t		ΔS_t		ΔO_t	
ECT_{t-1}	-0.0010	(0.0014)	-0.0075*	(0.0037)	-0.0243***	(0.0059)
ΔH_{t-1}	0.1632**	(0.0544)	0.6513***	(0.1466)	0.6314**	(0.2290)
ΔH_{t-2}	0.1427*	(0.0563)	0.1761	(0.1517)	0.3316	(0.2370)
ΔH_{t-3}	0.1505**	(0.0567)	0.0498	(0.1526)	0.0257	(0.2384)
ΔH_{t-4}	0.0530	(0.0557)	-0.4957**	(0.1500)	-0.5522*	(0.2343)
ΔH_{t-5}	0.0688	(0.0553)	0.0323	(0.1488)	0.1751	(0.2324)
ΔH_{t-6}	0.0545	(0.0541)	0.0814	(0.1457)	0.1379	(0.2276)
ΔS_{t-1}	0.0609.	(0.0320)	-0.0785	(0.0861)	0.6204***	(0.1346)
ΔS_{t-2}	0.0270	(0.0341)	-0.2764**	(0.0918)	0.3425*	(0.1434)
ΔS_{t-3}	0.1208***	(0.0360)	0.0070	(0.0968)	0.4527**	(0.1512)
ΔS_{t-4}	0.0785*	(0.0357)	-0.3387***	(0.0962)	-0.0893	(0.1503)
ΔS_{t-5}	0.0693*	(0.0337)	-0.1362	(0.0906)	0.1576	(0.1416)
ΔS_{t-6}	0.0234	(0.0288)	-0.1260	(0.0777)	-0.1820	(0.1213)
ΔO_{t-1}	-0.0053	(0.0223)	0.1634**	(0.0601)	-0.4554***	(0.0939)
ΔO_{t-2}	0.0135	(0.0251)	0.1596*	(0.0676)	-0.3400**	(0.1057)
ΔO_{t-3}	-0.0325	(0.0264)	0.1714*	(0.0710)	-0.1437	(0.1109)
ΔO_{t-4}	-0.0159	(0.0260)	0.1510*	(0.0700)	-0.0743	(0.1094)
ΔO_{t-5}	-0.0175	(0.0235)	0.1944**	(0.0634)	0.0361	(0.0990)
ΔO_{t-6}	-0.0051	(0.0188)	0.1294*	(0.0507)	0.1011	(0.0792)
N	357		357		357	
Adj. R^2	0.4516		0.1506		0.234	

Note: $*** p < 0.001; ** p < 0.01; * p < 0.05; . p < 0.10$

3.4 Impulse-Response Functions

A convenient feature of VECM models is that they can be readily converted to a cointegrated VAR (CVAR) in levels (Lütkepohl, 2005; Kilian & Lütkepohl, 2017; Hamilton, 2020). Through this, it is possible to have general impulse-response functions in levels to evaluate how unit shocks from our three core variables of interest respond to one another. While there is some persistence of orders in the inventories equation described in Table 5, it is possible such a relationship may be nothing more than spurious (and even so, its persistence is at a level that is statistically insignificant). Figure 3 provides standard responses to unit shocks within our trivariate VECM system.

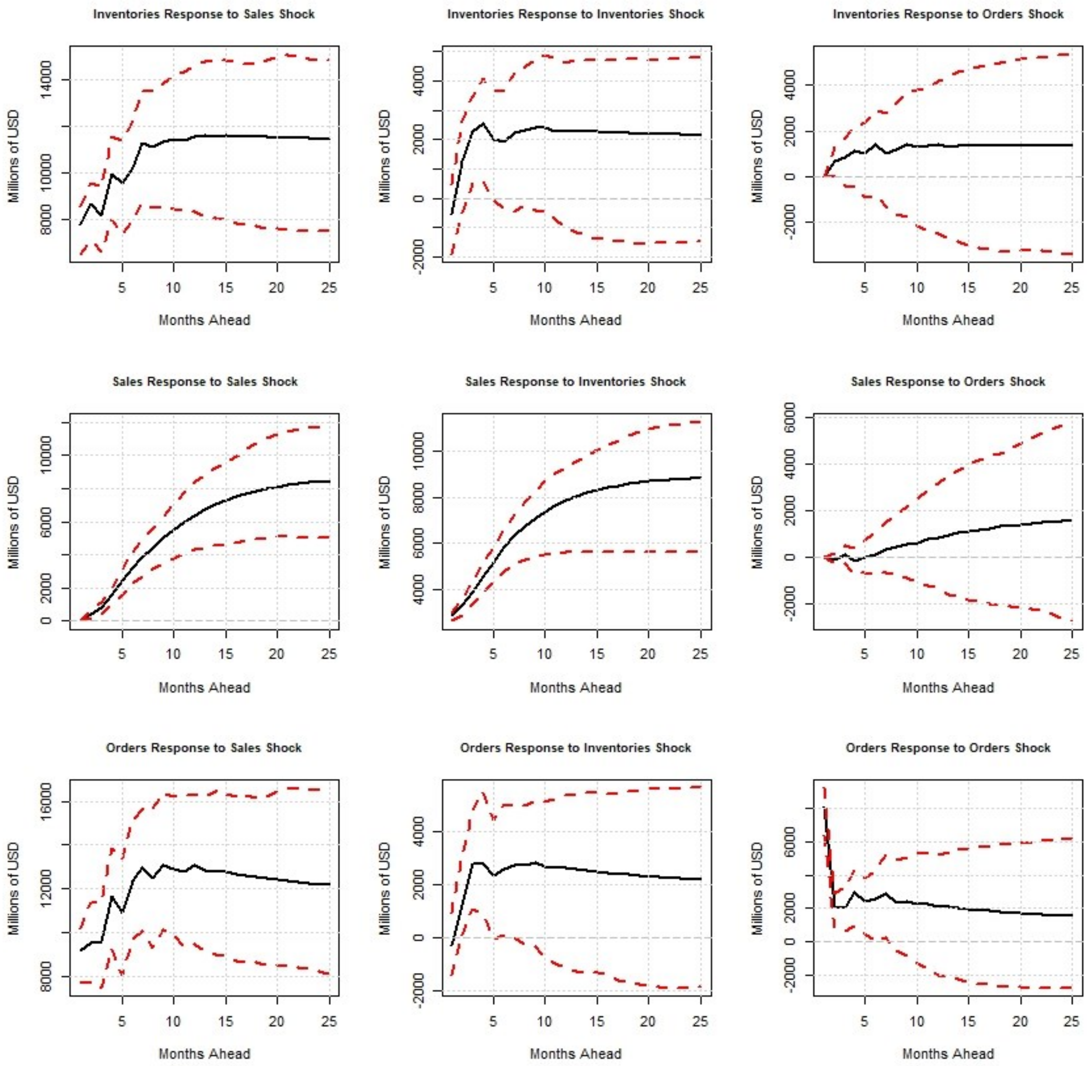

Figure 3. CVAR Impulse-Response Functions

Theoretically it is sensible to understand why inventories do not respond strongly to order shocks (as echoed in Table 5): the firm should be incentivized to build-to expected orders, therefore if firms have an unbiased model of shipments conditional on orders, their impact on inventories should be neutral at most. In contrast, however, unexpected shipments could signal sales volatility, and provide reason to hold higher levels of inventories (as the LQ model would predict). Inventory growth leading order growth Table 5 is more likely spurious given the results present in Figure 3, but nevertheless provides some utility for the prospected and interested forecaster.

3.5 Forecast Error Variance Decomposition

A final exercise of benefit to forecasters, and practitioners is that of a forecast error variance decomposition (FEVD). An FEVD can provide some insights on the volatility of forecasts associated with our variables of interest. These results for a forecast horizon of $t = 1 \dots t = 9$

periods (3 quarters) are described below in Table 6.

Table 6. FEVD Results

	Inventories (H_t)			Shipments (S_t)			Orders (O_t)		
Horizon	H_t	S_t	O_t	H_t	S_t	O_t	H_t	S_t	O_t
$t+1$	1.00	0.00	0.00	0.00	0.99	0.00	0.00	0.56	0.44
$t+2$	0.99	0.01	0.00	0.01	0.98	0.00	0.01	0.71	0.28
$t+3$	0.98	0.02	0.00	0.03	0.96	0.01	0.03	0.76	0.21
$t+4$	0.94	0.06	0.00	0.04	0.95	0.01	0.03	0.80	0.16
$t+5$	0.90	0.10	0.00	0.04	0.95	0.01	0.04	0.83	0.14
$t+6$	0.85	0.15	0.00	0.04	0.95	0.01	0.04	0.84	0.12
$t+7$	0.82	0.18	0.00	0.04	0.95	0.01	0.04	0.86	0.11
$t+8$	0.79	0.21	0.00	0.04	0.95	0.01	0.04	0.87	0.09
$t+9$	0.73	0.24	0.00	0.04	0.95	0.01	0.04	0.88	0.08

As we can see, the level of inventories explains most of its own forecast volatility, however, in later forecast horizons, sales can explain almost 25% of variation in inventory levels. This is consistent with the high persistence of inventories, as noted in Ramey & West (1999). Similarly, variation in shipments forecasts over long horizons are mostly explained by their own time series dynamics.

For orders, we see its variation is well-explained by its own dynamics, but best explained by variation in shipments, accounting for almost 90% of its variation in later forecast periods. This reinforces the importance of sales or shipments and its predictive content for inventories over orders. However, if orders are a valid proxy for expected demand, and forecasters are interested in forecasting such expectations, it is reliable, and advisable to use a combination of past demand expectations (orders), and shipments to do so.

4. Discussion

For total manufacturing, our results show that orders adjust at a rate that is both stronger and significant when compared to adjustment speeds for inventories and sales. This implies that deviations from the long-run in expected demand tend to revert to back to steady-state more meaningfully than realized demand.

The asymmetries in the adjustment rates between expected and realized demand can result in inventory accumulation. In a sense, the inclusion of orders in our model helps explain the slow adjustment puzzle of inventories by disentangling expected demand from realized sales. While we posit that our variation on the LQ model is superior to the baseline, traditional model, we can also test improvements in the goodness-of-fit between both single-equation models using a standard ANOVA test comparing our baseline equation and equation (18). Table 7 shows the results of this exercise.

Table 7. ANOVA Test Results

Model	Res. DF	RSS	DF	SSR	F-Stat	P-Value
Equation (18)	359	5030401936	--	--	--	--
Baseline Eq.	358	4833039644	1	197362292	14.619	0.00***
Note: $*** p < 0.001; ** p < 0.01; * p < 0.05; . p < 0.10$						

As we can see, there is a strong and statistically significant improvement in the fit of our reduced form LQ model compared to the existing baseline, reinforcing our proposition that orders do well to better explain the long-run relationships between inventories and sales implied by the LQ model, and are an integral component of the underlying data generating process for sales, despite orders lackluster predictive content.

Linking these findings to the literature, it becomes clear that this work contributes towards addressing several puzzles pervasive in the literature. As noted in the Maccini et al. (2009), two of the most prevalent puzzles in the inventories literature are the variance ratio puzzle, and the slow adjustment puzzle. The variance ratio puzzle states that despite firms operating under convex costs to smooth production, production volatility should be less than sales volatility, however, the data does not bear this outcome—as noted in Table 1, the volatility of Q_t is still larger than S_t. The slow adjustment puzzle states that the estimated stock-adjustment speeds of inventories are exceedingly slow relative to how wide inventory swings can be day-to-day for most manufacturers.

We offer some possible explanations within the context of our results. When evaluating potential sources of the variance puzzle, we offer the possibility that the Bullwhip Effect or demand amplification ($var(O_t)/var(S_t)$) can explain this. One possible reason production volatility likely exceeds sales volatility is that new orders generated from downstream firms signal higher levels of production than reflected in actual demand for upstream manufacturers.

Lee et al. (1997) offers several compelling explanations for this phenomenon, including: long lead times, competing forecasts, order batching costs, correlated ordering, price volatility, and shortage gaming. Future research would do well to focus on tying the key contributions from the operations research, and management science fields to economics to further expound on this specific puzzle.

Regarding adjustment speeds, it is possible that in instances of disequilibrium, inventories are not adjusting, rather orders (expected demand) are. In essence, early demand signals generated by new orders prompt stock adjustment indirectly, however, should actual shipments or sales be incongruent with orders, inventories will be created incidentally.

In other words, orders signal expected demand, wherein unsold finished goods (inventories) are generated when there is a positive difference between expected demand and realized demand ($O_t - S_t > 0$). This is just one plausible explanation; however, verification of this conjecture would require a more rigorous structural model beyond the scope of this work.

5. Concluding Remarks

We augment the existing LQ model with orders and find that the inclusion of orders has a significant impact in a reduced form setting. We find that the omission of orders biases the sign of the sales coefficient. Furthermore, we find asymmetry in the adjustment speeds between sales and orders in a conventional VECM setting. These results suggest that expected demand adjusts at a rate statistically stronger and faster than realized demand. This finding implies that part of the inventory adjustment speed puzzle can be explained by incongruence between expected and realized demand. When comparing the goodness-of-fit of our single-equation inventory investment equations, we find that there is a strong and statistically significant improvement in the fit our model with orders compared to the traditional solution without orders.

We stress that the LQ model is a parsimonious partial equilibrium representation of the representative firm's inventory investment decision; however, it is not without criticism and drawbacks. Of note, its fit compared to the underlying data generating process (DGP) is not particularly compelling. Furthermore, as with any structural model, its structural identification informs its reduced form identification, thus, the representative firm's choice of inventory investment is guided by our setup described by equations (1) through (5).

Works like West & Wilcox (1994), and Hamilton (2002) also do well to point out the limitations of the LQ model, in particular its weak global fit, and assumptions of cointegration between inventories, and sales, of which evidence is mixed within the literature. These critiques notwithstanding, the LQ model still serves as the backbone of the inventories literature and has been used as the starting point for studies outside of inventories such as the case of oil shocks to production in Herrera (2018).

Inventories and their relationship to the business cycle and its volatility, particularly considering growing woes in global supply chains, is still of interest to practitioners and policymakers alike and has motivated more recent pieces such as Gortz & Gunn (2018), Gortz et al (2022), Camacho et al. (2011), Crouzet & Oh (2016), Jones & Tuzel (2013) and Boileau & Letendre (2011). As interest in inventories continues to rebound from its once mature state, the field must be diligent in ensuring the assumptions of the structural equations that the literature is built upon are also updated and considerate of new information available to practitioners and researchers alike.

We argue that this paper serves as one such Pareto improvement to the benchmark LQ model by allowing for expected demand and realized demand to be disentangled from one another with minimal deviation from the setup of the original model described by Holt (1960) and Ramey & West (1999). These efforts provide new evidence towards solving the pervasive variance ratio and adjustment speed puzzles present in the literature.

While more structurally complex models such as Ramey (1989) and Humphreys et al. (2001) exist, their foundations are, too, informed by the LQ model, thus, it is critical for future research to consider updates to more complex general equilibrium models by distinguishing between expected and realized demand explicitly. Finally, applied researchers utilizing the LQ would be misguided by its original and albeit dated solution given the bias associated with the shipments term, in particular.

Statement on Reproducibility

All tables, figures, and regression estimates are produced using the software R. The script or code is available upon request.

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Notes

Note 1. This is calculated as the average of the Bureau of Economic Analysis (BEA) time series "INVCMRMT" (real manufacturing and trade inventories) divided by "GDPC1" (real GDP).

Note 2. This is the "baseline" solution to the LQ model.

Note 3. Throughout this paper, we will use "sales" and "shipments" interchangeably.

Note 4. This is driven by the well-known correlation between inventories and lagged output as well as the correlation between inventories and firm sales (around 90.4% in the data utilized in this study).

Note 5. Other pieces looking at the implications of the Bullwhip Effect in terms of profitability and forecast precision are explored in Metters (1997), and Bray & Mendelson (2013), among others.

Note 6. This is done to interpret our data in real terms, or in quantities, rather than nominal terms.

Note 7. See Jiang et al. (2021) for more details on this phenomenon.

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The data that support the findings of this study are available on request from the corresponding author. The data are not publicly available due to privacy or ethical restrictions.

Data sharing statement

No additional data are available.