

# Visualization of Algebraic Identities in $\mathbb{R} \times \mathbb{R} \times \mathbb{R} = \mathbb{R}^3$ (in Space) at Secondary Education

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## Abstract

Algebraic identities, which are considered as one of the important subjects of mathematics and mathematics education, have important areas of use such as physics, chemistry, many branches of engineering, as well as calculating the distances among the earth, sun and stars and planning space travels. Algebraic identities play an important role in the mathematics curriculum and in mathematics in general. However, it is seen as one of the subjects that are difficult to learn by students in formal education. In this context, this study aims to convey the visual proofs of the identities to the researchers in an explanatory way by using the visualization approach in order to make it easier for students to understand some of the important identities in  $\mathbb{R}^3$ . In the teaching of proofs of algebraic identities in secondary education, teachers only give formulas or make algebraic proofs in lessons rather than showing visual proofs and relating them to daily life. This situation poses a problem for students with visual learning tendencies. For this reason, it is thought that showing the proofs of identities by making visual models or visualizations in lessons can be a solution to this problem. In this study, document analysis technique was used within the scope of descriptive analysis and it was designed as a compilation study. Visual proofs of some identities were made by using squares and rectangles from geometric shapes. In this context, in the light of the study, some suggestions were made to guide interested researchers in this regard.

**Keywords:** Algebraic identities, Visual proof, Visualization

## 1. Introduction

Mathematics is one of the scientific fields that makes significant contribution to the development of science and technology in the 21st century. Mathematics is the essence of science just as the air and water in life. It is necessary to train educators who know mathematics and apply it to daily life and transfer knowledge to new generations. There are also a number of new methods and techniques to train these educators and enable them to use

visualization-based approaches in lessons. The visualization approach, which is one of these teaching methods and techniques, has an important place in explaining abstract mathematical concepts. In order to increase the retention of concepts by providing meaningful learning, it is important to use visual shapes or models in lessons, through computer technology to make algebraic and visual proofs of identities. It is important that students learn by questioning the cause-effect relationship rather than rote learning. In order to understand these relations better, the "proof" at the core of mathematics is important. Proof is seen as one of the mathematical skills that are difficult to learn and teach. At the same time, proving skills play an important role in mathematics and mathematics education. The method of proving is of great importance for mathematicians, because human beings act from the idea of accepting the truth of anything they have seen. Reviewing the literature, there are various definitions of proof.

Proof means demonstrating the correctness of a situation or case with evidence, testing and checking its accuracy (Hersch, 1993). Demonstrating the truth or falsity of a mathematical proposition or theorem is the main purpose of a mathematical proof (Hanna, 2000). Demonstrating the accuracy of data obtained from experiments and observations by using scientific methods is the common feature of all sciences. By making use of this feature, the correctness of a mathematical proposition or theorem is shown by mathematicians by making use of mathematical definitions, axioms and logical rules (Hanna & Barbeau, 2002; Polster, 2004). According to Turkish Language Association (TDK), proof is defined as "providing, proving its truthfulness or accuracy by showing evidence or putting forward evidence". Similarly, the word "prove" is considered synonymous with 1) to verify 2) to demonstrate, (TDK, 2020). A mathematical proof is expressed as "demonstrating the correctness of a mathematical proposition or theorem based on the reasons and this is accepted by mathematicians" (Dede, 2013). Alsina and Nelsen (2006) expressed in their study that when students are asked to prove basic identities, they usually show them in numerical values or prove them algebraically.

This indicates that the students learn the formulas by rote, and it ensures that the information is forgotten in a short time and creates difficulties for later learning. It is thought that it would be more appropriate to use visual presentations or proof of concepts instead of making students memorize abstract mathematical concepts (Baki, 2008). Many students do not know why proving is important in lectures. However, proof plays an important role in learning mathematics in a meaningful way (Tekin & Konyalıoğlu, 2010).

Identities, which make an important contribution to the solution of the problems encountered in daily lives, are seen as one of the important sub-learning areas of the secondary school mathematics curriculum. Rather than grasping the visual and algebraic proof of basic identities in  $\mathbb{R}^3$ , students often memorize and apply them to the problems they encounter. In this context, visual learning and techniques that appeal to the sense organs are easier to learn and provide permanent learning (Tekin & Konyalıoğlu, 2010).

In their study, Barth and Demirtaş (1997) emphasize that the more the teachers explain their mathematics lessons using visual models and concrete examples, the easier it will be to teach

the concepts and increase the academic achievement of the students in a positive way (Budanur, 2004). In similar studies, it has been stated that the use of visualization approach in mathematics lessons increases the academic achievement of students positively (Aktaş, 2006; Budanur, 2004; Demirel, 1996; Eritici et al., 2005). Teachers' presentation of the lessons using visualization approach and visual models will enable students to see details better and increase the quality of teaching identities (Özdemir, Duru, & Akgün, 2005).

Due to the abstract nature of mathematical concepts, students had difficulty in understanding the concepts. Concrete and visual materials used in solving this problem played an important role in concretizing or visualizing concepts (Baş, 2021). It is important that the concrete materials should be designed in accordance with course objectives and daily updates. According to the Ministry of National Education (2018), it was evaluated that teachers' use of concrete materials in the lessons in the renewed mathematics curriculum contributed positively to the development of psychomotor behaviors of students.

A study by the Ministry of National Education (MEB) revealed that teachers who used equipment, visual models, computer animations and educational tools in the lessons increased their student success significantly and learning became permanent (Şan, 2008). In their study, Yenilmez and Şan (2008) also found that the solution of mathematical concepts and problems was easier to understand through the visual tools used in the lessons. They drew attention to the importance of the visualization approach, depending on the conclusion that the students were insufficient in the interpretation of geometric shapes.

Özer and Şan (2013) stated in their study that it would be appropriate to make algebraic and visual proof of abstract identities in order to enable students to be active in mathematics lessons. In addition, due to the use of algebraic expressions while solving mathematical problems, it is important to teach the subject of identities to explain the concepts such as geometric figures' length, area, volume. Ziegler and Stern (2014) stated in their study that algebraic identities and expressions served as a key in forming the basis of abstract algebra. In another study conducted by Jupri and Drijvers (2016), students had difficulty in transferring algebraic expressions while solving problems related to daily life.

In their study, Shin and Bryant (2015) emphasized that the use of algebraic and visual representations such as diagrams, figures and shapes in the teaching of fractions made abstract mathematical concepts easier to understand. Algebraic identities are very important in forming the basis for mathematics and other courses from secondary education to university (Muchoko et al., 2019).

Birgin and Demirören (2020) conducted a study with 180 secondary school students attending the seventh and eighth grades in Afyonkarahisar province. They revealed that students had problems comprehending simple visual and algebraic expressions because teachers gave priority to numerical operations in algebraic expressions. Students also stated that they could not write visual and algebraic expressions in different types of representations.

Algebraic identities, one of the subjects that are believed to be difficult to learn in secondary

education, are widely used in solving some problems encountered in daily lives in mathematics, physics, space science, engineering and technology. Students encounter identities for the first time in secondary school 7th grade textbooks, and they are also encountered in high school and university education. It is used in many learning areas taught in secondary education such as square roots, factorization, simplification, polynomials, trigonometry, limit, derivative, matrix and determinants, integral. Identities also play an important role in forming the basis of the algebra course. In this study, the visualization of the algebraic identities in  $R^3$  at the secondary level was made in order to enable students to learn the abstract algebra concepts more easily. In this direction, together with the demonstration of algebraic proof of algebraic identities, it is aimed to introduce other researchers the visual proof of identities through geometric figures to facilitate learning.

## 2. Method

This study, which was carried out within the scope of qualitative research, used document analysis technique as a descriptive analysis model and was also designed as a compilation study. In this study; the document analysis technique was used to examine the written materials in detail and systematically (Yıldırım & Şimşek, 2021). Document analysis includes the analysis of written materials given or presented as documents, pictures, papers, books, films, videos, e-books, photographs or electronic files that show proof of something (Yıldırım & Şimşek, 2021).

## 3. Visualization

Demonstrations, visual models and visualizations have an important place in understanding the mathematics lesson easily, which is seen as a course of abstract concepts. Polya, one of the important scientists, states that it is important to draw a figure first in solving the problem by establishing the relationship between visualization and problem solving. Because drawing the shape suitable for the structure of the problem will make the problem easier to understand and solve. Visualization has various purposes in teaching mathematics. Dufour-Janvier et al. (1987) summarized their study as follows (Özgün-Koca, 1998).

In the essence of mathematics, there are visual representations.

- Visual representations play an important role in embodying a concept in various ways.
- Visual representations are used to overcome some difficulties in mathematics lessons.
- It is useful to use visual representations to make mathematics more interesting and to enable students to have a positive attitude towards the lesson.

Concretization, which is seen as one of the teaching principles, uses the sense organs to convey information with figures, pictures, diagrams or models to provide visualization in teaching. Visualization has an active role in making an object or an action visible. Visualization is a method that makes it easier to understand a concept or situation that cannot be seen or described by naked eye (Bagni, 1998; McCormick et al., 1987).

Stylianou (2002) investigated the relationship between visualization and problem solving skills. Stylianou stated that a good mathematician effectively used visual tools or models such as diagrams, graphs and figures while solving problems.

In the process of solving any problem of a person, visualizing visual tools such as pictures, diagrams, tables and graphics in their minds or using reasoning and algebraic methods is also considered as a stage of the visual solution method (Çilingir Altınır & Önal, 2022).

Visualization has an effective place in concretizing mathematical concepts (Arcavi, 20003). When the relevant literature is examined, it is understood that the studies using visualization approach in mathematics lessons increase the academic success of the students positively (Eroğlu, 2006; Konyalıoğlu, 2003).

Visualization is defined in a more comprehensive way as follows. "Developing previously unknown thoughts; to think in line with these thoughts and to improve understanding by using pictures, figures, diagrams. It is the process and ability of transforming, interpreting and communicating information by using visual models (Cited by Tekin, 2010)."

Visualization is described as the ability to create, interpret, use and reflect pictures, images, diagrams on paper by using technological tools or a pen in order to describe a concept or information, to explain it to others, to think and to develop it (Uysal-Koğ & Başer, 2012).

In a study conducted with primary school teachers, it was emphasized that the use of visualization approach in lessons made the lesson more enjoyable and easier to learn (Bayar & Zengin, 2020).

#### **4. Algebraic Identities and Visual Proofs**

Expressions that are always true for every nonzero real number value of equality are called identity. Algebraic identities are widely used in solving many problems encountered in daily lives, especially in mathematics and science. While teaching algebraic identities at secondary level, the visualization approach and the use of multiple representations are important in terms of ensuring the intelligibility of the concepts.

Algebraic identities are recognized as an important set of formulas that are widely used in the secondary school mathematics curriculum and in mathematics in general. They form the basic principle of algebra and help to perform calculations in simple and easy steps. Knowing and recognizing algebraic identities help students learn mathematical concepts. It will also enable them to develop their procedural knowledge skills while applying these operations algebraically. The main problem for the students when learning and applying identities is that the majority of them tend to memorize them completely. It is thought that approaches such as using computer animations, visual models or representations can be effective in preventing this. Identities, which are also used in algebra course, have extensive usage in many fields and industries, as they help to solve many real life problems. In addition, algebraic identities are used in modeling many problems. Students taking the algebra course are introduced to many concepts such as algebraic expressions and solving equations, the properties of equality and inequalities in addition to many other topics. Algebraic identities are defined as equations

in which the value of the left side of the equation is equal to the right side equally for all real number values of the variables. These identities contain constants and variables on both sides of the equation. Unlike equations, algebraic identities are true for all real number values of variables. Visual representations and visualization have an important role for students to learn the abstract algebraic identities more meaningfully and easily. Visual proofs have started to be seriously considered as an alternative to traditional proofs, and the use of computer facilities in the visualization of some concepts has gained significant value for the last 20 years. It is important why visual representations are used while visualizing. Visual (non-verbal) proofs (Proof Without Words, PWWs) are defined as mathematical expressions that result in figures, diagrams and graphs to prove a certain mathematical proposition and theorem (Nelsen, 2000). For example, visual or nonverbal proofs; do not contain any words other than real or numeric symbols and geometric drawings. There is debate as to whether a nonverbal proof is qualified as a proof.

In their study, Barth and Demirtaş (1997) emphasized that it was important to use concrete examples or visual materials in lessons in order to learn abstract concepts more easily and meaningfully during education.

In algebraic proofs, there are steps of operation and logical inferences showing the accuracy of these steps (Borwein & Jögenon, 2001). Borwein and Jögenon (2001) state in their study that a good visual proof must satisfy three conditions. These are;

- **Reliability:** the proof of a theorem must be acceptable to everyone. When proofs are made by different people, contradictory results should not occur.
- **Consistency:** For a proposition to be considered valid and true, it must be consistent first. When the proof of any theorem is shown by different people, the truth-values of the proofs must be the same.
- **Repeatability:** The proof should be performed by different people and its accuracy should be shown.

Although the three criteria mentioned are also valid for algebraic proofs, the above features should also be considered in the visual proof process. As Bardelle (2009) states, visual proofs are proofs that are not explained by any interpretation, that are not conveyed directly to the reader in written form using concrete and visual models, and that the reader is expected to construct the proof effectively. Visual proofs should be formed by the reader using only numbers, letter expressions, arrows, dots, diagrams, figures, pictures and related symbols, without using or interpreting a verbal expression (Bardelle, 2009). Visual proof must have the characteristics of algebraic proof.

Visual representations and visual proofs have an important place in teaching concepts and theorems in mathematics and mathematics education. In this context, a number of mathematicians and logicians state that the use of visual representations can have the highest possible benefits, especially in mathematical proofs (Brown, 1999; Davis, 1993; Giaquinto, 1992, 2005; Mancosu, 2005). A visual (non-verbal) proof can be thought as a ‘proof’ made by using visual representations, pictures or other visual tools to demonstrate the correctness of a



mathematical idea, proposition, equation or theorem (Casselman, 2000). Such proofs can be considered elegant rather than formally or mathematically precise because they are clear and visible. In another study, he stated that students learnt abstract concepts better and the learning environment was enriched because of using well-designed teaching materials in the classroom environment (Yalın, 2001). When the literature is reviewed, visual proofs have been used as an alternative proof method instead of traditional proofs for the last 20 years.

Today, there is an ongoing debate about whether visual proofs that are up to date are important or not. Despite these debates, many researchers state that the use of pictures, figures, diagrams, graphics and other visual materials in the lessons is beneficial and students can learn the lessons more easily (Hanna & Sidoli, 2007). Using a pen, you can draw signs, shapes, pictures and diagrams about concepts on paper. Visualization is considered as a process as it requires the creation, interpretation and reflection of the product that emerges during the drawing phase. It is stated that the final shape that emerges can be evaluated as a product of visualization (Gierdien, 2007). In the proof process, visual representations have an important place as an integral part of the proof. While some researchers state that visual representations are important in proofs, some other researchers claim that shapes, diagrams, pictures, and other visual representations have a limited but important role in proofs (Hanna & Sidoli, 2007).

In MEB councils, it has been stated that the use of concrete and visual materials stimulating more than one sensory organ in the lessons during the education process positively affects student achievement and the retention of knowledge (Şan, 2008). Visual presentations or visual proofs play an important role in teaching abstract mathematical concepts (Tekin & Konyalıoğlu, 2010). 90 % of visually learned information is remembered, and approximately 65 % of people in a community tend to learn visually (McCoy, 2019).

In a study conducted with university students, Kim et al. (2006) stated that the number of students who learnt visually was 80 % more than the number of students who learnt verbally. It is important to use visual proofs and visualization approach in teaching abstract mathematical concepts for students who tend to learn visually.

Traditionally, in Turkish education system, some students learn mathematical formulas by rote and have difficulty remembering the formulas when they need. As Martin (2020) pointed out, the saying “A shape is worth a thousand words” drew attention to the importance of visualization in mathematics teaching. Methods, techniques, strategy, visual models and visual proofs used in teaching identities in Turkey and abroad have been examined in detail. Algebraic and visual proofs of identities are shown in a simplified way for students to understand based on the data collected. In addition, a teaching model including these proofs was proposed. Visual proofs or visual representations of all of the identities given below can be used in paper, cardboard, wood, Styrofoam, metal sheet, computer animations, etc.

The visual proofs of the algebraic identities given below in  $R^3$  will help us understand and grasp each identity visually better. Algebraic and visual proofs of each of the basic algebraic identities in  $R^3$  are shown respectively.

4.1  $(x + y)^3 = x^3 + 3x^2 \cdot y + 3x \cdot y^2 + y^3$  *Visual Proof of Identity*

Proof:

First, let's create a cube with a side length of  $(x + y)$  unit (Figure 1).

Eight geometric shapes, two different cubes and prisms were obtained by drawing the line segments EF and GH, whose side length would be  $y$  unit from the N corner of the cube (Figures 1 and 1a). Volume of cube with side length  $(x + y)$  unit (in Figure 1) was calculated as,

$$V_{(x+y)} = (x + y) \cdot (x + y) \cdot (x + y) = (x + y)^3 \cdot br^3$$

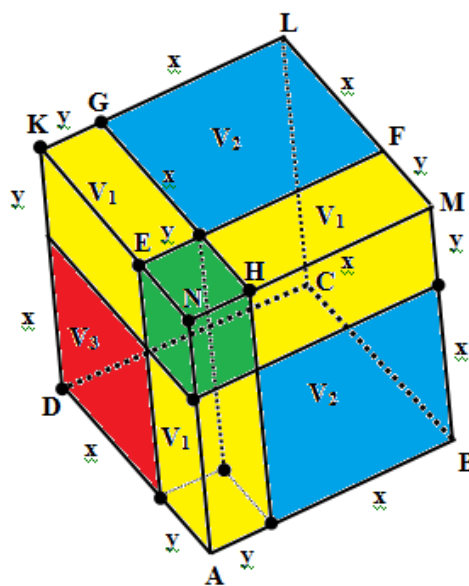


Figure 1. Visual proof of the identity  $(x + y)^3 = x^3 + 3x^2 \cdot y + 3x \cdot y^2 + y^3$

Geometric figures with eight parts were obtained from Figure 1 as seen in Figure 1a.



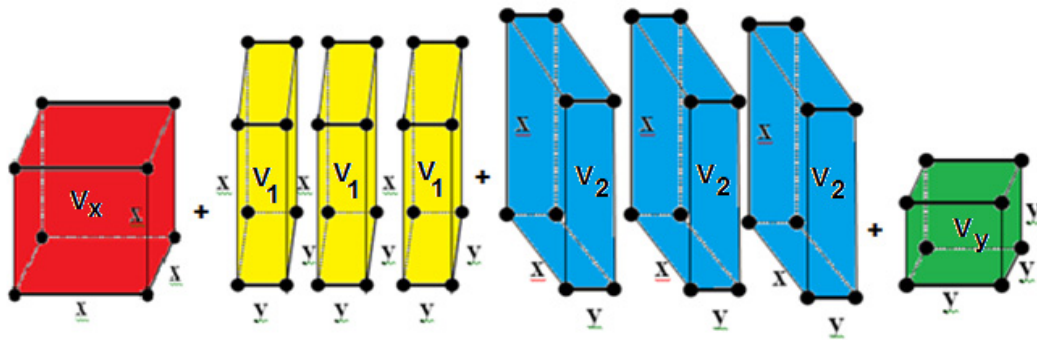


Figure 1a. Visual representation of geometric figures obtained from a cube with a side  $(x + y)$  unit

Let's find the volumes of the prisms with volume  $V_x$ ,  $V_1$ ,  $V_2$  and  $V_y$  given in Figure 1a, respectively.

$$V_x = x \cdot x \cdot x = x^3 \cdot br^3$$

$$V_1 = y \cdot y \cdot x = y^2 \cdot x \cdot br^3$$

$$V_2 = x \cdot y \cdot x = x^2 \cdot y \cdot br^3$$

$$V_y = y \cdot y \cdot y = y^3 \cdot br^3$$

If the volumes added up, it was found;

$V_T = V_x + 3 \cdot V_1 + 3 \cdot V_2 + V_y = x^3 + 3y^2 \cdot x + 3x^2 \cdot y + y^3$ . As a result, the total volume of the prisms in Figure 1a was equal to the volume of the cube with side length  $(x + y)$  unit. In this case; by applying the distributive property of multiplication over addition, this could be written as

$V_{(x+y)} = (x + y) \cdot (x + y) \cdot (x + y) = (x + y)^3 = x^3 + 3x^2 \cdot y + 3x \cdot y^2 + y^3$ . So, visual proof was completed.

#### 4.1a The Algebraic Proof of Identity $(x + y)^3 = x^3 + 3x^2 \cdot y + 3x \cdot y^2 + y^3$

Proof:

The algebraic proof of  $(x + y)^3$  was shown as step by step.

$$(x + y)^3 \text{ could be written as } (x + y) \cdot (x + y) \cdot (x + y) = (x + y) \cdot (x + y)^2 = (x + y) \cdot (x^2 + 2xy + y^2)$$

Applying the distributive property of multiplication over addition; then

$$(x + y)^3 = x \cdot (x^2 + 2xy + y^2) + y \cdot (x^2 + 2xy + y^2) = x^3 + 2x^2 \cdot y + x \cdot y^2 + y \cdot x^2 + 2x \cdot y^2 + y^3. \text{ From this,}$$

As a result,  $(x + y)^3 = x^3 + 3x^2 \cdot y + 3x \cdot y^2 + y^3$  was proved as algebraically.

4.2 The Visual Another Proof of the Identity  $(x + y)^3 = x^3 + 3x^2 \cdot y + 3x \cdot y^2 + y^3$  (Figures 2 and 2a)

Proof:

Let's create two different prisms with  $x$  and  $y$  unit heights from the cube whose side length is  $(x + y)$  unit as seen in Figure 2. The volume of a cube with side  $(x + y)$  unit is,

$$V_{(x+y)} = (x + y) \cdot (x + y) \cdot (x + y) = (x + y)^3 \cdot br^3 \text{ (as seen Figure 2).}$$

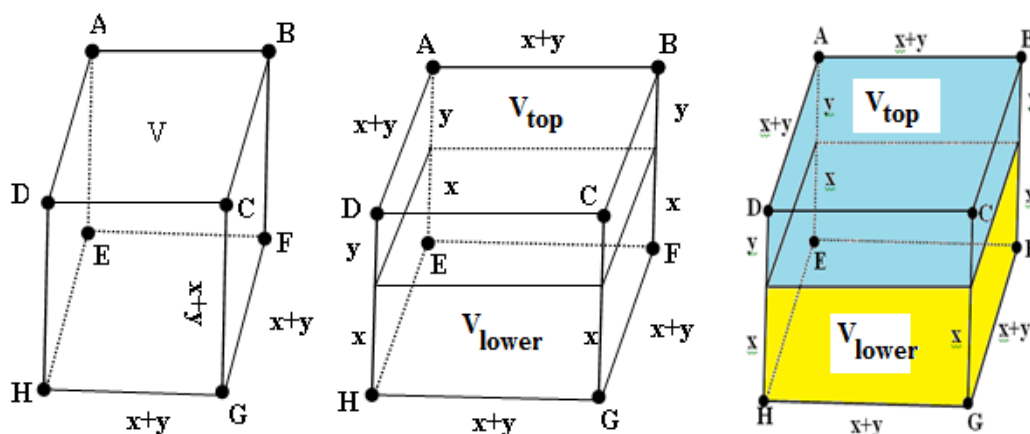


Figure 2. Visual representation of prisms of  $x$  and  $y$  height formed from the cube with side length  $(x + y)$  unit

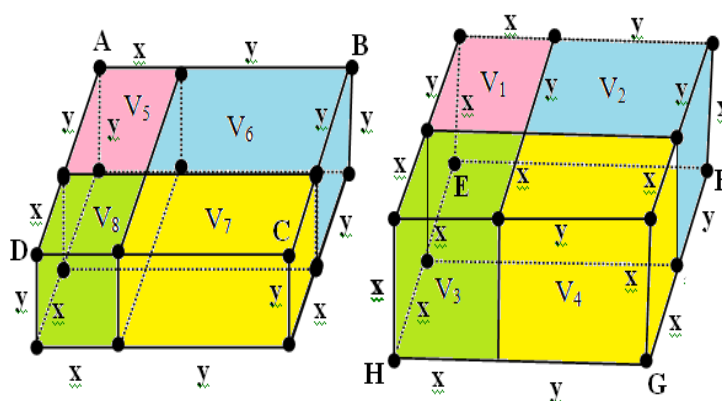


Figure 2a. Visual representation of geometric figures obtained from a cube with a side  $(x + y)$  unit

The volumes of the prisms were respectively obtained from Figure 2. As seen in Figure 2a.

So it can be written algebraically the total volume of prisms as,  $V_{\text{total}} = V_{\text{top}} + V_{\text{lower}}$

Then, to find the volume of each prism in Figure 2a, the values of prisms are put respectively

in the above algebraic formula.

$$V_{\text{top}} = V_5 + V_6 + V_7 + V_8 = x \cdot y \cdot y + y \cdot y \cdot y + x \cdot y \cdot y + x \cdot x \cdot y = 2x \cdot y^2 + x^2 \cdot y + y^3$$

$$V_{\text{lower}} = V_1 + V_2 + V_3 + V_4 = x \cdot y \cdot x + y \cdot y \cdot x + x \cdot x \cdot x + x \cdot y \cdot x = 2x^2 \cdot y + x \cdot y^2 + x^3$$

$$V_{\text{total}} = V_{\text{top}} + V_{\text{lower}} = 2x \cdot y^2 + x^2 \cdot y + y^3 + 2x^2 \cdot y + x \cdot y^2 + x^3 = (x^3 + 3x^2 \cdot y + 3x \cdot y^2 + y^3).$$

This volume of  $V_{\text{total}}$  is equal to the volume of a cube with side length  $(x + y)$  unit. Therefore, It could be written as,  $V_{(x+y)} = V_{\text{total}} = V_{\text{top}} + V_{\text{lower}}$

$V_{(x+y)} = (x + y) \cdot (x + y) \cdot (x + y) = (x + y)^3 = (x^3 + 3x^2 \cdot y + 3x \cdot y^2 + y^3)$  was obtained. So, visual proof was completed.

#### 4.3 Visual Proof of the Identity $(x - y)^3 = x^3 - 3x^2y + 3xy^2 - y^3$ (Figures 3, 3a, and 3b).

Let's create a cube at corner C of a cube with a side length of  $x$  unit, with a side length of  $(x - y)$  unit (Figure 3). The created  $V_1$  volume cube; The volumes of the prisms with  $V_3$  volume on the left side,  $V_2$  volume behind and  $V_{\text{lower}}$  volume below were calculated respectively (Figure 3a). Constructing a cube with side length  $(x - y)$  unit from the side with  $x$  unit cube as seen in figure 3. The volume of a cube with side length  $(x - y)$  unit was found as,

$$V_{(x-y)} = (x - y) \cdot (x - y) \cdot (x - y) = (x - y)^3 \cdot br^3$$

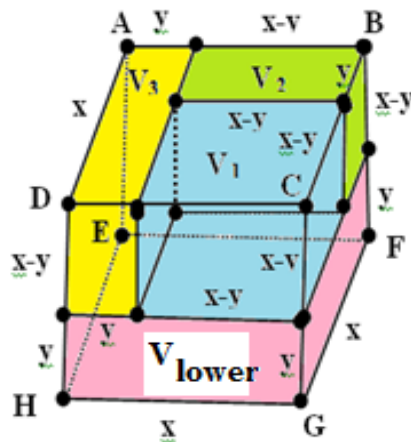


Figure 3. Visual representation of different geometric shapes derived from  $x$  unit cubes

The volumes of the prisms with volumes  $V_1$ ,  $V_2$ ,  $V_3$  and  $V_{\text{lower}}$  are respectively separated from Figure 3 as seen in Figure 3a.

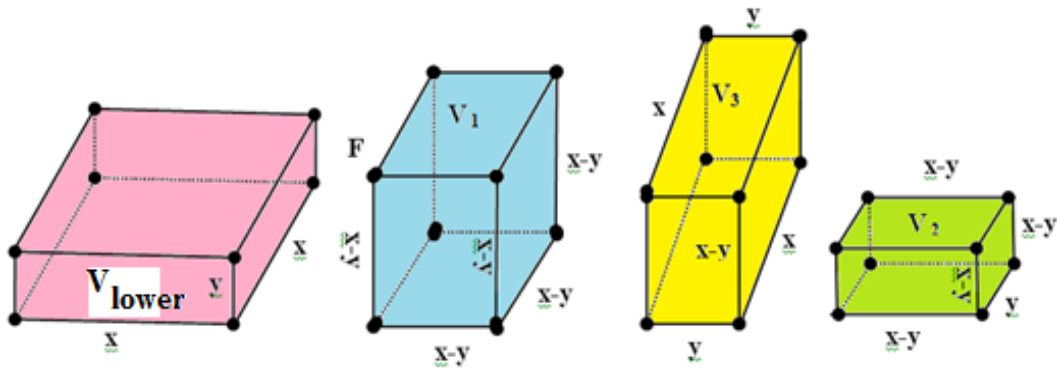


Figure 3a. Visual representation of the volumes of the prisms with volumes  $V_{lower}$ ,  $V_1$ ,  $V_2$  and  $V_3$

The volumes of the prisms with volumes  $V_1$ ,  $V_2$ ,  $V_3$  and  $V_{lower}$  obtained from  $x$  unit cube are calculated respectively.

$$V_{lower} = x \cdot x \cdot y = x^2 \cdot y \cdot br^3$$

$$V_1 = (x - y) \cdot (x - y) \cdot (x - y) = (x - y)^3 \cdot br^3$$

$$V_2 = y \cdot (x - y) \cdot (x - y) = y \cdot (x - y)^2 = y \cdot (x^2 - 2x \cdot y + y^2) = (x^2 \cdot y - 2x \cdot y^2 + y^3) \cdot br^3$$

$$V_3 = x \cdot y \cdot (x - y) = (x^2 \cdot y - x \cdot y^2) \cdot br^3$$

The volume of the cube with side length  $(x - y)$  unit was found by subtracting the sum of the volumes of the prisms with  $V_2$ ,  $V_3$  and  $V_{lower}$  volumes from the volume of the cube with side length  $x$  unit as seen in Figure 3b.

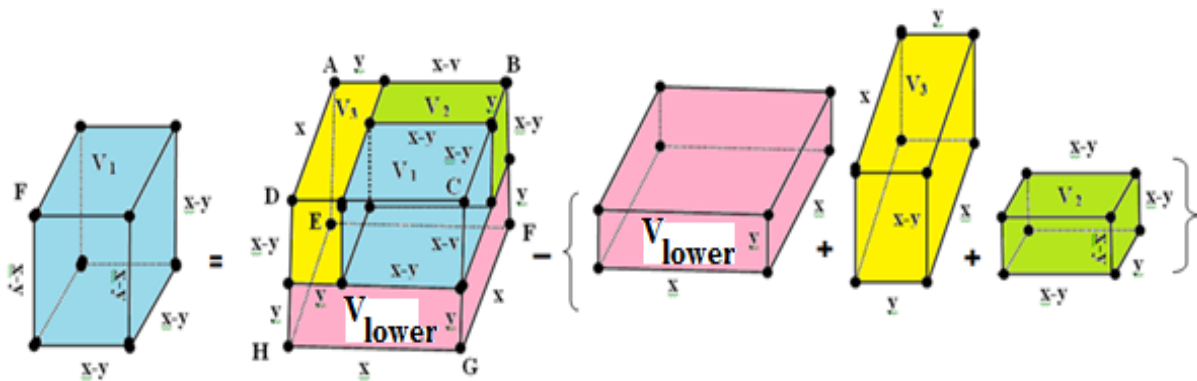


Figure 3b. Visual expression of the volume of the cube with side length  $(x - y)$  unit

This explanation could be written algebraically as,

$$V_{(x-y)} = V_x - [V_{lower} + V_3 + V_2] = x \cdot x \cdot x - [x \cdot x \cdot y + x \cdot y \cdot (x - y) + y \cdot (x - y) \cdot (x - y)]$$

This algebraic formula can be written as,

$$V_{(x-y)} = x^3 - [x^2 \cdot y + x^2 \cdot y - x \cdot y^2 + y \cdot (x^2 - 2xy + y^2)] = x^3 - (2x^2 \cdot y - x \cdot y^2 + y \cdot x^2 - 2x \cdot y^2 + y^3)$$

$$V_{(x-y)} = x^3 - (3x^2 \cdot y - 3x \cdot y^2 + y^3) = x^3 - 3x^2 \cdot y + 3x \cdot y^2 - y^3$$

The volume of a cube with side length  $(x - y)$  unit was found as,

$V_{(x-y)} = (x - y) \cdot (x - y) \cdot (x - y) = (x - y)^3 \cdot br^3$  at the beginning in Figure 3. So, this volume is equal to,

$$V_{(x-y)} = V_x - [V_{\text{lower}} + V_3 + V_2].$$

As a result,  $V_{(x-y)} = (x - y) \cdot (x - y) \cdot (x - y) = (x - y)^3 = (x^3 - 3x^2 \cdot y + 3x \cdot y^2 - y^3)$  was completed visually.

#### 4.3a Algebraic Proof of the Identity $(x - y)^3 = x^3 - 3x^2 \cdot y + 3x \cdot y^2 - y^3$

Proof:

The algebraic proof is respectively shown as step by step.

The algebraic formula  $(x - y)^3$  could be written as

$$(x - y) \cdot (x - y) \cdot (x - y) = (x - y) \cdot (x - y)^2 = (x - y) \cdot (x^2 - 2xy + y^2)$$

Applying the distributive property of multiplication over addition and subtraction.

$$(x + y)^3 = x \cdot (x^2 - 2xy + y^2) - y \cdot (x^2 - 2xy + y^2) = x^3 - 2x^2 \cdot y + x \cdot y^2 - y \cdot x^2 + 2x \cdot y^2 - y^3$$

$$(x - y)^3 = x^3 - 3x^2 \cdot y + 3x \cdot y^2 - y^3$$

As a result,  $(x - y)^3 = x^3 - 3x^2 \cdot y + 3x \cdot y^2 - y^3$  was proved as algebraically.

#### 4.4 Visual Proof of the Identity $x^3 - y^3 = (x - y) \cdot (x^2 + x \cdot y + y^2)$ (Figures 4 and 4a)

Let's subtract a cube with a side length of  $y$  unit from one corner from the volume of a cube with a side length of  $x$  unit as seen in Figure 4.

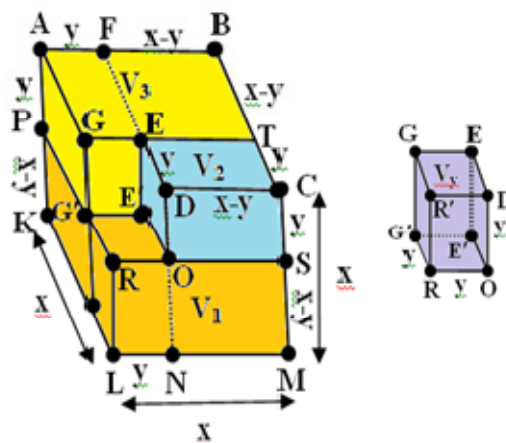


Figure 4. Visual representation of cubes with side lengths  $x$  unit and  $y$  unit

The volume of cubes with side lengths  $x$  unit and  $y$  unit are calculated as,

$$V_x = x \cdot x \cdot x = x^3 \text{ br}^3$$

$$V_y = y \cdot y \cdot y = y^3 \text{ br}^3$$

The volume of the shape remaining is calculated after discarding the cube with side length  $y$  unit as seen in Figure 4.

$$V_k = V_x - V_y = (x^3 - y^3) \cdot \text{br}^3$$

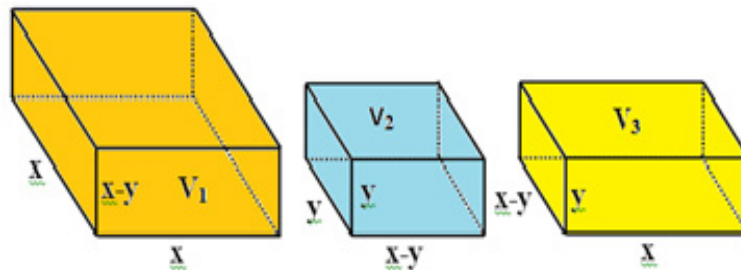


Figure 4a. Visual representation of the geometric figures remaining

The remaining volumes of the  $V_1$ ,  $V_2$  and  $V_3$  given in Figure 4a are found respectively.

$$V_1 = x \cdot x \cdot (x - y) = x^2 \cdot (x - y) \cdot \text{br}^3$$

$$V_2 = y \cdot (x - y) \cdot y = y^2 \cdot (x - y) \cdot \text{br}^3$$

$$V_3 = x \cdot (x - y) \cdot y = x \cdot y \cdot (x - y) \cdot \text{br}^3$$

By summing up the volumes found as

$$V_T = V_1 + V_2 + V_3 = x^2 \cdot (x - y) + y^2 \cdot (x - y) + x \cdot y \cdot (x - y) = (x - y) \cdot (x^2 + x \cdot y + y^2)$$

This volume  $V_T = V_1 + V_2 + V_3$  is equal to the volume of the shape remaining calculated at the beginning in Figure 4.

$$V_k = V_x - V_y = (x^3 - y^3) = V_T = V_1 + V_2 + V_3 = (x - y) \cdot (x^2 + x \cdot y + y^2) \text{ was shown visually.}$$

#### 4.4a Algebraic Proof of the Identity $x^3 - y^3 = (x - y) \cdot (x^2 + x \cdot y + y^2)$

Proof:

The algebraic proof of  $(x^3 - y^3)$  was respectively shown as step by step.

The algebraic formula  $x^3 - y^3$  could be written as,

$$x^3 - y^3 = (x - y) \cdot (x^2 + x \cdot y + y^2) = x \cdot (x^2 + x \cdot y + y^2) - y \cdot (x^2 + x \cdot y + y^2)$$

Applying the distributive property of multiplication over addition and subtraction. Then,

$$x^3 - y^3 = x \cdot (x^2 + x \cdot y + y^2) - y \cdot (x^2 + x \cdot y + y^2) = x^3 + x^2 \cdot y + x \cdot y^2 - y \cdot x^2 - x \cdot y^2 - y^3 = x^3 - y^3$$



As a result,  $(x - y)(x^2 + xy + y^2) = x^3 - y^3$  was proved as algebraically.

#### 4.5 Visual Proof of the Identity $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$

Proof:

Let's create two different cubes with side lengths  $x$  unit and  $y$  unit as seen in Figure 5.

When the small cube with a side length of  $y$  unit was put on the big cube the  $x$  side, the total volume of the geometric shape formed was found as  $V_{(x+y)} = x^3 + y^3 \cdot br^3$ .

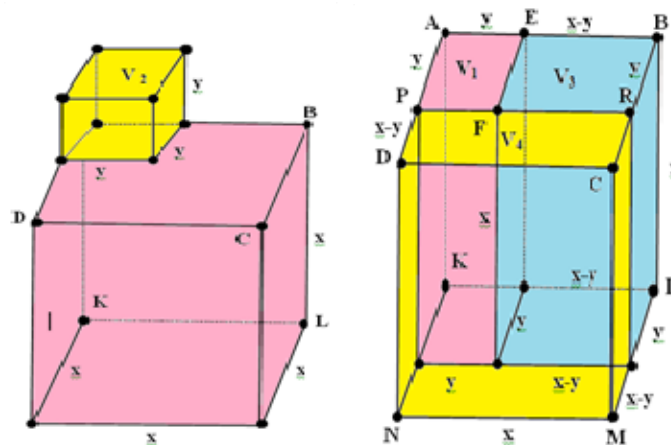


Figure 5. The visual representation of two cubes, the small cube with a side length of  $y$  unit on the big cube the  $x$  side

Let's create three different prisms by drawing the line segments  $PR$  and  $EF$ , whose side length will be  $y$  unit from the corner  $A$  of the big cube, after placing the cube with  $y$  unit on the cube with side length  $x$  unit as seen on right Figure 5.

Then, let's separate the formed geometric shapes and find the volume of each one respectively in Figure 5a.

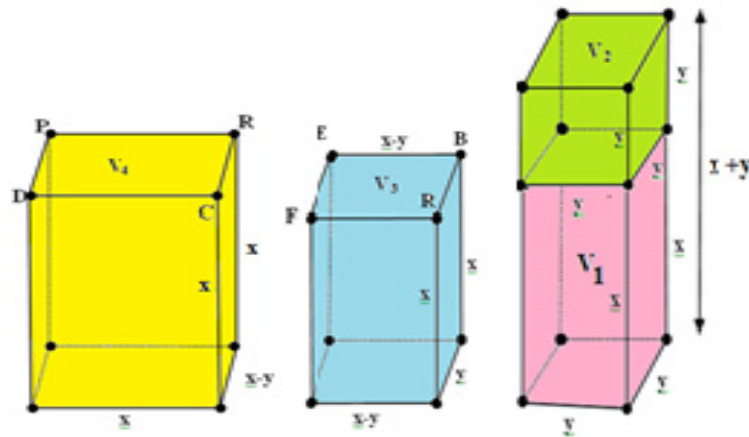


Figure 5a. Visual representation of prisms which are separated from Figure 5 on the right

Let's find the volume of each one prism.

The general formula of any prism is  $V_T = (\text{The base area of prism} \times \text{The height of prism})/3$

This formula used each one prism can be calculated respectively as

$$V_1 = x \cdot (x - y) \cdot x = x^2 \cdot (x - y) = (x^3 - x^2 \cdot y) \cdot br^3$$

$$V_2 = y \cdot (x - y) \cdot x = (x^2 \cdot y - x \cdot y^2) \cdot br^3$$

$V_{(x+y)} = V_1 + V_2 = y \cdot y \cdot (x + y) = (x \cdot y^2 + y^3) \cdot br^3$ . Therefore, by adding  $V_1$ ,  $V_2$  and  $V_{(x+y)}$  to obtain

$$V_T = V_1 + V_2 + V_{(x+y)} = (x^3 - x^2 \cdot y) + (x^2 \cdot y - x \cdot y^2) + (x \cdot y^2 + y^3) \cdot br^3$$

$$V_T = V_1 + V_2 + V_{(x+y)} = x \cdot (x^2 - xy + y^2) + y \cdot (x^2 - xy + y^2) = (x^2 - xy + y^2) \cdot (x + y) \cdot br^3$$

The total volume as  $V_{(x+y)}$  of the geometric shape formed at the beginning in Figure 5 is equal to  $V_T = V_1 + V_2 + V_{(x+y)}$  in Figure 5a.

This explanation could be written algebraically as  $V_{(x+y)} = x^3 + y^3 = (x^2 - xy + y^2) \cdot (x + y)$

As a result;  $x^3 + y^3 = (x^2 - xy + y^2) \cdot (x + y)$  was shown visually.

#### 4.5a Algebraic Proof of the Identity $x^3 + y^3 = (x + y) \cdot (x^2 - x \cdot y + y^2)$

Proof:

The algebraic proof was respectively shown as step by step. The algebraic formula,

$(x^3 + y^3)$  can be written as,  $x^3 + y^3 = (x + y) \cdot (x^2 - x \cdot y + y^2)$ . After that, applying the distributive property of multiplication over addition and subtraction.

$$x^3 + y^3 = x \cdot (x^2 - x \cdot y + y^2) + y \cdot (x^2 - x \cdot y + y^2)$$

$$(x + y) \cdot (x^2 - x \cdot y + y^2) = x^3 - x^2 \cdot y + x \cdot y^2 + y \cdot x^2 - x \cdot y^2 + y^3 = x^3 + y^3$$

As a result,  $(x + y) \cdot (x^2 - x \cdot y + y^2) = (x^3 + y^3)$  was proved as algebraically.

## 6. Conclusion and Suggestions

In this study, visual proofs and algebraic proofs of the formulas of algebraic identities in three dimensions (in space) are made. For this purpose, it is thought that the visual and algebraic proofs of the basic identities are given in detail and guide the researchers who are interested in this subject. Students may reach wrong conclusions by applying mathematical rules incorrectly or by falling into misconceptions during the stages of proving algebraic identities. The main reason why they face such a situation is that they do not fully understand the form of proof and they have difficulty in understanding the relationships between abstract mathematical concepts (Arcavi, 2003; Duval, 2002).

The use of visual shapes in teaching mathematical concepts, solving problems and proving some theorems make it easier for students with learning difficulties to understand (Suh & Moyer, 2007). Kolloffel et al. (2009) state that there are many diagrams that provide visualization of mathematical concepts. Therefore, in the concretization of mathematical concepts, figures, pictures, diagrams, visual models, computer animations and graphic organizers play an important role.

In another study conducted with pre-service teachers, it was stated that visual proofs in mathematics lessons attracted students' attention, showed positive attitudes towards the lesson, and facilitated the understanding of students with learning difficulties (Demircioğlu & Polat, 2015). It is stated that it is important to use visual shapes in the teaching of algebraic expressions in order to make the transition from numerical representations to algebraic representation easier while solving some problems in the first stage of primary education (Birgin & Demirören, 2020).

In the light of the above information, In order for visualization to be effective, teachers must explain the definitions and properties of the concepts to the students verbally and in writing first. Then it will be useful for them to draw a figure related to the concept and make the necessary explanations on the figure.

Visual proofs play an important role in establishing the relationship between conceptual knowledge and procedural knowledge. For this reason, it is recommended that mathematics teachers frequently include visual proofs in their lessons.

It is thought that it would be appropriate to make algebraic proofs of different theorems in the lessons in order to enable students to develop their analytical thinking and reasoning skills in a positive way. Thus, students will be able to access the formula more easily. Instead of making students memorize mathematical formulas, showing where and how the formulas are obtained and making the proofs of theorems or formulas by drawing visual forms will provide permanent learning.

It is thought that it would be appropriate to include the visualization and visual proofs of theorems or concepts in the lessons in accordance with the cognitive levels of the students so that the students do not have difficulty in proving process. It can also be helpful to get the

opinions of teachers and students about visual proof. The use of algebra tiles, computer animations or visual models in teaching the subject of identities can be effective in terms of concretizing and visualizing the concepts.

As a result, in teaching the concepts of algebraic identities, which are widely used in mathematics and science, it would be appropriate for teachers to give more importance to visual figures and proofs in lessons.

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